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An Application of Duality Theory to Control-Approximation Problems

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In this paper we give an approach to the solution of control-approximation problems using duality theory. We solve the dual problem by homogeneous programming and apply the method to optimal control of distributed parameter systems. Our discussion of the implementation contains numerical examples.

1. INTRODUCTION

We consider a class of control problems which can be considered as approximation problems, where the family of approximating functions depends linearly on an infinite-dimensional parameter space. The parameter is constrained to the unit ball of the space. A complicating factor from the numerical point of view is the representation of the approximating functions by infinite series. The aim of this paper is to give a method where these infinite-dimensional optimization problems are replaced by finite-dimensional problems and to show how to estimate effectively the optimal value of the original problem.

Investigations in this direction have been undertaken in [6, 13]. Dualizing the original problem we obtain a convex optimization problem, where the variables lie in the unit ball of an infinite-dimensional space. For a parabolic boundary control problem, it is shown in [5] that by minimizing over some particular finite-dimensional subspaces, the dual problem converts into a finite-dimensional optimization problem. By this procedure lower bounds on the optimal value are obtained. We study more general operators for which this procedure can be applied.

The finite-dimensional optimization problem, which includes a constraint, can be replaced by an unconstrained problem using the homogeneity of the objective and constraint. This method can be applied to elliptic, parabolic and hyperbolic control problems, considering initial-, boundary- and

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distributed control with different observations. In the case of a problem in Hilbert space the unconstrained minimization does not have to be carried out. It is shown that the solution of a nonlinear equation in one real variable is sufficient to compute lower bounds. Furthermore we prove that for each lower bound we can determine a control which is a solution of an approximation of the primal problem and obtain an upper bound which is "best" in a certain sense. By increasing the dimension of the finite-dimensional subspace we show a decreasing behavior of the upper and an increasing behavior of the lower bound such that we have a monotone sequence of inclusions for the optimal value. In the last section we give a numerical example and discuss different types of control problems.

2. Application of Duality Theory

Let X and Y be normed linear spaces, $U \subseteq X$ a nonvoid convex set, $R: X \to Y$ a continuous linear operator and $z \in Y$ fixed. We consider the problem

$$\inf_{u \in U} \|Ru - z\|_Y = \rho. \tag{P1}$$

For existence theorems on this problem we refer, e.g., to [1]. Denote with B_{Y^*} the unit ball of the dual space Y^*.

Then $(P1)$ is equivalent to

$$\inf_{u \in U} \max_{l \in B_{Y^*}} l(Ru - z) = \rho. \tag{P2}$$

By duality theory [cf. 3, Sect. 19, Theorem b], we obtain

$$\sup_{l \in B_{Y^*}} [\inf_{u \in U} l(Ru) - l(z)] = \rho. \tag{D1}$$

Since it is often advisable for numerical purposes to replace the dual space Y^* by a weakly* closed linear subspace W, we consider instead of $(D1)$

$$\sup_{l \in B_W} [\inf_{u \in U} l(Ru) - l(z)] = \delta, \tag{D2}$$

where $B_W = \{w \in W \subseteq Y^*: \|w\|_{Y^*} \leqslant 1\}$. Calling $R^*: Y^* \to X^*$ the adjoint operator of R, then

$$\sup_{l \in B_W} [\inf_{u \in U} (R^* l) u - l(z)] \leqslant \rho. \tag{D2}$$

If we assume that $U = B_X$, the unit ball in X, then we simplify $(D2)$,

$$\sup_{l \in B_W} [-\|R^*l\|_{X^*} + l(z)] \leqslant \rho. \tag{$D2$}$$

The map d: $Y^* \to \mathbb{R}$ defined by

$$d(l) = \|R^*l\|_{X^*} - l(z) \tag{1}$$

is weakly* lower semicontinuous. Since B_W is weakly* compact in Y^*, we know that the supremum in $(D2)$ (and also in $(D1)$) is attained,

$$\max_{l \in B_W} [-\|R^*l\|_{X^*} + l(z)] = -\min_{l \in B_W} [\|R^*l\|_{X^*} - l(z)] \leqslant \rho.$$

Hence $(D2)$ is a constrained minimization problem,

$$\text{minimize } d(l) \text{ subject to } l \in W, \qquad \|l\|_{Y^*} \leqslant 1,$$

where d is given by (1).

We observe that $d(\cdot)$ is a positive homogeneous functional on Y^* of degree 1 as well as the constraint functional $\|\cdot\|_{Y^*}$.

THEOREM 1. *For arbitrarily chosen, but fixed, $\lambda > 0$ the problem*

$$\textit{minimize } d(l) + \lambda \|l\|^2_{Y^*} \textit{ subject to } l \in W \tag{$\bar{D}$}$$

has a solution $\bar{l} \in W$ with optimal value $\bar{\delta} = d(\bar{l}) + \lambda \|\bar{l}\|^2_{Y^*}$. In the case $\bar{l} = \theta$, the solution $\hat{l}$ of $(D2)$ is also given by $\hat{l} = \theta$. Otherwise $\hat{l} = \bar{l}/\|\bar{l}\|_{Y^*}$ solves $(D2)$ and the optimal value δ of $(D2)$ is determined by $\delta = -(\bar{\delta} - \lambda \|\bar{l}\|^2_{Y^*})/\|\bar{l}\|_{Y^*}$.

Proof. By definition of d,

$$d(l) + \lambda \|l\|^2_{Y^*} \geqslant -\|l\|_{Y^*} \|z\|_Y + \lambda \|l\|^2_{Y^*} > 0$$

for $\|l\|_{Y^*} > \|z\|_Y/\lambda$. Therefore $(\bar{D})$ is equivalent to

$$\text{minimize } [d(l) + \lambda \|l\|^2_{Y^*}] \text{ subject to } \|l\|_{Y^*} \leqslant \|z\|_Y/\lambda, \qquad l \in W,$$

which is a minimization of a weakly* lower semicontinuous functional on a weakly* compact set, hence the minimum is attained. Let $\bar{l} \in W$ be a solution of $(\bar{D})$,

$$d(\bar{l}) + \lambda \|\bar{l}\|^2_{Y^*} \leqslant d(l) + \lambda \|l\|^2_{Y^*} \qquad \text{for all } l \in W. \tag{2}$$

Assume $\bar{l}=\theta$ and $d(l^*)=\|R^*l^*\|_{X^*}-(l^*,z)<0$ for some $l^*\in B_W$. Then we obtain, for $\varepsilon>0$ small enough,

$$d(\varepsilon l^*)+\lambda\,\|\varepsilon l^*\|_{Y^*}^2=\varepsilon(d(l^*)+\varepsilon\lambda\,\|l^*\|_{Y^*}^2)<0,$$

a contradiction to the optimality of $\bar{l}$ for $(\bar{D})$. Assuming $\bar{l}\neq\theta$, (2) implies, with $\lambda>0$,

$$d(\bar{l})\leqslant d(l) \qquad \text{for all } l\in W, \|l\|_{Y^*}\leqslant\|\bar{l}\|_{Y^*},$$

and

$$d(\bar{l})\leqslant d(l) \qquad \text{for all } l\in W, \|l\|_{Y^*}/\|\bar{l}\|_{Y^*}\leqslant 1.$$

Put $m=l/\|\bar{l}\|_{Y^*}$, $\bar{m}=\bar{l}/\|\bar{l}\|_{Y^*}$, then for $\|m\|_{Y^*}\leqslant 1$

$$\|\bar{l}\|_{Y^*}\,d(\bar{m})=d(\bar{m}\,\|\bar{l}\|_{Y^*})\leqslant d(m\,\|\bar{l}\|_{Y^*})=\|\bar{l}\|_{Y^*}\,d(m),$$

or

$$d(\bar{m})\leqslant d(m) \qquad \text{for all } m\in B_W.$$

This shows that $\bar{l}/\|\bar{l}\|_{Y^*}$ solves $(D2)$ and

$$-\delta=d(\bar{m})=d(\bar{l})/\|\bar{l}\|_{Y^*}=(\bar{\delta}-\|\bar{l}\|_{Y^*}^2)/\|\bar{l}\|_{Y^*}.$$

The proof of Theorem 1 relies heavily on homogeneous programming (cf. [2]).

Concerning the calculation of upper bounds we refer to the case $W=Y^*$, where, using the duality theorem, we have for the optimal point $\bar{u}$

$$\begin{aligned}-\bar{l}(R\bar{u})+\bar{l}(z)&\leqslant\|R\bar{u}-z\|_Y=-\|R^*\bar{l}\|_{X^*}+\bar{l}(z)\\&\leqslant-(R^*\bar{l})\bar{u}+\bar{l}(z)=-\bar{l}(R\bar{u})+\bar{l}(z).\end{aligned} \tag{3}$$

Hence in (3) equality holds, which implies

$$\|R^*\bar{l}\|_{X^*}=(R^*\bar{l})\bar{u}. \tag{4}$$

Equation (4) will be used to select $\bar{u}\in B_{Y^*}$ for computing an upper bound $\|R\bar{u}-z\|_Y$.

3. Minimal Systems

In control problems with distributed parameter systems many operators R can be expressed in the following form: Let $\{y_i\}\subseteq Y$, $\{x_i^*\}\subseteq X^*$ and R be defined by

$$Ru = \sum_{i=1}^{\infty} c_i x_i^*(u) y_i,$$

assuming that R is linear and continuous. The adjoint is given by

$$R^* l = \sum_{i=1}^{\infty} c_i l(y_i) x_i^*.$$

We shall make use of the following definition: A set of functions $\{y_i\}_{\mathbb{N}} \subseteq Y$ is called *minimal* if and only if for each $j \in \mathbb{N}$ we have $y_j \notin cl\, \text{span}\{y_i \mid i \in \mathbb{N}, j \neq i\}$. The minimality can be characterized:

THEOREM 2 [4, p. 264]. *The family $\{y_i\}_{\mathbb{N}} \subseteq Y$ is minimal if and only if there exists a biorthogonal sequence $\{y_i^*\}_{\mathbb{N}} \subseteq Y^*$ such that*

$$y_j^*(y_i) = \delta_{ji}, \qquad i, j \in \mathbb{N}.$$

Under the assumption that $\{y_i\}$ are minimal we define

$$W = \text{span}\{y_1^*, ..., y_N^*\}$$

and hence the problems $(\bar{D})$ and $(D2)$ become finite-dimensional optimization problems

$$\min_{l \in W}(\|R^* l\|_{X^*} - l(z) + \lambda \, \|l\|_{Y^*}^2)$$

$$= \min_{a \in \mathbb{R}^N} \left\| \sum_{i=1}^{N} a_i c_i x_i^* \right\|_{X^*} - \sum_{i=1}^{N} a_i y_i^*(z) + \lambda \left\| \sum_{i=1}^{N} a_i y_i^* \right\|_{Y^*}^2 = -\bar{\delta}_N, \quad (\bar{D}_N)$$

$$\min_{a \in \mathbb{R}^N} \left\| \sum_{i=1}^{N} a_i c_i x_i^* \right\|_{X^*} - \sum_{i=1}^{N} a_i y_i^*(z) = -\delta_N, \qquad \left\| \sum_{i=1}^{N} a_i y_i^* \right\|_{Y^*} \leqslant 1. \quad (D_N 2)$$

LEMMA 3. *Let $\{y_i\}$ be minimal and the biorthogonal system $\{y_i^*\}$ be complete in Y^*. Then δ_N is a monotonically increasing sequence of numbers with* $\lim_{N \to \infty} \delta_N = \delta$.

On the other hand, if $\{x_i^*\}$ are minimal, then we have the existence of a biorthonormal set $\{x_i\}$ in X^{**} and assuming $\{x_i\} \subseteq X$ we can define in the standard way a finite-dimensional problem for obtaining upper bounds,

$$\text{minimize} \left\| \sum_{i=1}^{N} b_i c_i y_i - z \right\|_Y \text{ subject to} \left\| \sum_{i=1}^{N} b_i x_i \right\|_X \leqslant 1. \qquad (P_N)$$

In the case where X and Y are Hilbert spaces and $\{x_i\}$, $\{y_i\}$ are orthonormal, we have the following connection between $(D_N 2)$ and (P_N):

THEOREM 4. *If* $(\hat{a}_1,\ldots,\hat{a}_N) \neq \theta_N \in \mathbb{R}^N$ *solves* $(\bar{D}_N)$, *then* $d^{-1}(c_1\hat{a}_1,\ldots, c_N\hat{a}_N) \in \mathbb{R}^N$, *with* $d^2 = \sum_{i=1}^N c_i^2\hat{a}_i^2$ *is a solution of* (P_N). *Furthermore*, $-\delta_N$ *and* ρ_N, *denoting the optimal values of* $(D_N 2)$ *and* (P_N), *respectively, have the inclusion property*

$$\delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \rho \leqslant \cdots \leqslant \rho_2 \leqslant \rho_1.$$

Proof. In this particular Hilbert space case $(\hat{b}_1,\ldots,\hat{b}_N)$ is optimal for (P_N), if and only if it solves the problem

$$\min \sum_{i=1}^N b_i^2 c_i^2 - 2\sum_{i=1}^N b_i c_i(y_i, z), \qquad \sum_{i=1}^N b_i^2 \leqslant 1,$$

or if and only if there exists a multiplier $\mu \geqslant 0$ such that

$$\sum_{i=1}^N \hat{b}_i^2 \leqslant 1 \qquad \text{and} \qquad \hat{b}_i c_i^2 - c_i(y_i, z) + \mu\hat{b}_i = 0, \qquad i = 1,\ldots, N. \tag{5}$$

On the other hand $(\hat{a}_1,\ldots,\hat{a}_N)$ solves $(\bar{D}_N)$ if and only if it minimizes, for some $\lambda > 0$,

$$\left[\sum_{i=1}^N a_i^2 c_i^2\right]^{1/2} - \sum_{i=1}^N a_i(y_i, z) + \lambda \sum_{i=1}^N a_i^2 \tag{6}$$

or equivalently is a solution of

$$d^2 = \sum_{i=1}^N \hat{a}_i^2 c_i^2, \qquad \hat{a}_i c_i^2 d^{-1} - (y_i, z) + 2\lambda\hat{a}_i = 0, \qquad i = 1,\ldots, N. \tag{7}$$

Multiplying (7) by c_i and using $\hat{b}_i = \hat{a}_i c_i d^{-1}$, we obtain (5), where the multiplier μ is given by $\mu = 2\lambda d > 0$. The inclusion property follows directly from the definition of $(D_N 2)$ and (P_N).

The relations $\lim_{N\to\infty} \delta_N = \rho$ and $\lim_{N\to\infty} \rho_N = \rho$ hold if $\{y_i\}$ and $\{x_i\}$ are dense in Y and X, resp. Under the same assumptions of Theorem 4 we can simplify the calculation of $(\hat{a}_1,\ldots,\hat{a}_N)$ as follows:

THEOREM 5. *The vector* $(\hat{a}_1,\ldots,\hat{a}_N) = \theta_N$ *is optimal for* $(\bar{D}_N)$ *if and only if* $\sum_{i=1}^N (y_i, z)^2 c_i^{-2} \leqslant 1$. *Otherwise let* $d > 0$ *be the unique solution of*

$$\sum_{i=1}^N ((y_i, z)\, c_i(2\lambda d + c_i^2)^{-1})^2 = 1, \tag{8}$$

then the optimal point $(\hat{a}_1,\ldots,\hat{a}_N)$ *is given by*

$$\hat{a}_i = (y_i, z)(2\lambda + d^{-1}c_i^2)^{-1}, \qquad i = 1,\ldots, N. \tag{9}$$

Proof. Observe that (6) is not differentiable at θ_N. Hence θ_N is optimal if and only if, for the directional derivatives of (6),

$$\left[\sum_{i=1}^{N} c_i^2 a_i^2\right]^{1/2} - \sum_{i=1}^{N} a_i(y_i, z) \geqslant 0 \qquad \text{for all } a \in \mathbb{R}^N.$$

This is equivalent to

$$\left[\sum_{i=1}^{N} \alpha_i^2\right]^{1/2} - \sum_{i=1}^{N} \alpha_i(y_i, z)\, c_i^{-1} \geqslant 0 \qquad \text{for all } \alpha \in \mathbb{R}^N,$$

or, using the Euclidean norm in $\mathbb{R}^N$,

$$\|\alpha\| - (\alpha, \eta) \geqslant 0 \qquad \text{for all } \alpha \in \mathbb{R}^N, \tag{10}$$

where $\eta = ((y_1, z)\, c_1^{-1}, \ldots, (y_N, z)\, c_N^{-1})$.

However, (10) holds if and only if $\|\eta\| \leqslant 1$. If $\|\eta\| > 1$, then the optimality conditions (7) hold. Solving (7) for $\hat{a}_i$ we obtain (9). Substituting (9) into the expression for d^2 we are led to (8). The unicity of $d > 0$ follows from the monotonicity of the left-hand side of Eq. (8).

Hence, if X is a Hilbert space and $\{x_i\}$ is orthonormal, then a lower bound can be calculated by computing a solution of Eq. (8). If the same requirement is met by Y and $\{y_i\}$, we obtain automatically an upper bound of the optimal value, which can be interpreted as the best for a certain approximate problem.

4. Several Applications in Distributed Control Systems

In this section, we represent generalized solutions of partial differential equations by Fourier series. For a justification we refer to the standard literature (see, e.g., [12, Sects. 29, 43, 44]).

Elliptic Control Problems

Let A be a second-order differential operator

$$A = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

with $a_{jk}, b_j, c \in L_\infty(\Omega)$, $i, j, k = 1, \ldots, n$, $\Omega \subseteq \mathbb{R}^n$ open, bounded and its boundary Γ a C^∞-hypersurface. Furthermore A is assumed to be formally self-adjoint and strongly elliptic, i.e., there exists $\gamma > 0$ with

$$\sum_{j,k=1}^{n} a_{jk}(x)\, x_j x_k \geqslant \gamma \sum_{j=1}^{n} x_j^2 \qquad \text{for all } x \in \mathbb{R}^n.$$

Boundary conditions are of the type

$$By(x) = \sum_{j,k=1}^{n} n_j(x)\, a_{jk}(x) \frac{\partial}{\partial x_k} y(x) + y(x).$$

For the distributed control problem

$$Ay(x) = u(x), \qquad x \in \Omega,$$
$$By(x) = 0, \qquad x \in \Gamma,$$

with $u \in L_2(\Omega)$ a generalized solution can be written as

$$y(x) = (Ru)(x) = \sum_{k=1}^{\infty} \mu_k^{-1}(u, e_k)\, e_k(x)$$

with $(u, e) = \int_\Omega u(x)\, e(x)\, dx$ and $\{\mu_k\}$, $\{e_k\}$ are the corresponding eigenvalues and eigenfunctions of the problem. The eigenfunctions have the property of being orthonormal in $L_2(\Omega)$. Hence choosing $Y = X = L_2(\Omega)$, $c_i = \mu_i^{-1}$, $x_i^* = y_i = e_i$, $i \in \mathbb{N}$, we can apply the results of Section 3, since an orthonormal system is minimal and identical with its biorthogonal system and furthermore $\{e_k\}$ is complete in $L_2(\Omega)$.

Parabolic Control Problems

First, we consider a system with control of the initial state:

$$y_t(t, x) - Ay(t, x) = 0, \qquad (t, x) \in (0, T) \times \Omega,$$
$$By(t, x) = 0, \qquad (t, x) \in (0, T] \times \Gamma,$$
$$y(0, x) = u(x), \qquad x \in \Omega.$$

A generalized solution for $u \in L_2(\Omega)$ can be defined by

$$y(t, x) = \sum_{k=1}^{\infty} \exp(-\mu_k^2 t)(u, e_k)\, e_k(x).$$

Observing the state at time $T > 0$, operator R is given by

$$Ru(x) = y(T, x) = \sum_{k=1}^{\infty} \exp(-\mu_k^2 T)(u, e_k)\, e_k(x),$$

and we have the same representation as in the previous part with $c_i = \exp(-\mu_i^2 T)$ and the same remarks on the applicability of Section 3 are valid.

If we observe the behavior of the temperature at a certain point $\bar{x} \in \Omega$, the operator R has to be defined as

$$Ru(t) = y(t, \bar{x}) = \sum_{k=1}^{\infty} e_k(\bar{x})(u, e_k) \exp(-\mu_k^2 t).$$

Hence we choose $X = L_2(\Omega)$, $Y = L_2(0, T)$, $c_i = e_i(\bar{x})$, $x_i^* = e_i$ and $y_i = \exp(-\mu_i^2)$. In the case of dimension $n = 1$ we know (cf. [11]) that $\{\exp(-\mu_k^2)\}$ form a minimal system and we can use the results of Section 3.

Investigating a system with boundary and distributed control

$$\begin{aligned} y_t(t,x) - Ay(t,x) &= u_g(t,x) = g(x)\,u(t), && (t,x) \in (0,T) \times \Omega, \\ By(t,x) &= v_h(t,x) = h(x)\,v(t), && (t,x) \in (0,T] \times \Gamma, \\ y(0,x) &= 0, && x \in \Omega, \end{aligned}$$

where the controls u_g, v_h are assumed to be separated in functions of space- and time-variables $g, h \in C(\bar{\Omega})$, $u, v \in L_2(0, T)$. We have, as a generalized solution at time $T > 0$,

$$\begin{aligned} R(u+v)(x) = y(T,x) = \sum_{k=1}^{\infty} \Bigg[&(g, e_k) \int_0^T \exp(-\mu_k^2(T-t))\,u(t)\,dt \\ &+ \langle h, e_k \rangle \int_0^T \exp(-\mu_k^2(T-t))\,v(t)\,dt \Bigg] e_k(x), \end{aligned}$$

with $\langle h, e_k \rangle = \int_\Gamma h(x)\,e_k(x)\,dx$. We select $X = L_2(0, T)^2$, $Y = L_2(\Omega)$ (for $n = 1$ or $h \equiv 0$), $c_i = 1$ and

$$x_i^* = ((g, e_i) \exp(-\mu_i^2(T - \cdot)), \langle h, e_i \rangle \exp(-\mu_i^2(T - \cdot)), \qquad i \in \mathbb{N}.$$

Hyperbolic Control Problems

In the following problems we observe the state at time $T > 0$. We start with controlling the initial state

$$\begin{aligned} y_{tt}(t,x) - Ay(t,x) &= 0, && (t,x) \in (0,T) \times \Omega, \\ By(t,x) &= 0, && (t,x) \in (0,T] \times \Gamma, \\ y(0,x) &= u(x), && x \in \Omega, \\ y_t(0,x) &= 0, && x \in \Omega, \end{aligned}$$

and obtain

$$Ru(x) = \sum_{k=1}^{\infty} \cos \mu_k T (u, e_k)\, e_k(x).$$

For the distributed control problem

$$\begin{aligned} y_{tt}(t,x) - Ay(t,x) &= g(x)\,u(t), && (t,x) \in (0,T) \times \Omega, \\ By(t,x) &= 0, && (t,x) \in (0,T] \times \Gamma, \\ y(0,x) &= 0, && x \in \Omega, \\ y_t(0,x) &= 0, && x \in \Omega, \end{aligned}$$

we define

$$Ru(x) = \sum_{k=1}^{\infty} (g, e_k) \int_0^T \mu_k^{-1} \sin(\mu_k(T-t))\, u(t)\, dt\, e_k(x),$$

$u \in L_2(0, T) = X$, $Y = L_2(\Omega)$. Conditions on the minimality of $\{\sin \mu_k \cdot\}$ can be found in [8] using results by [7] and [10].

5. Some Numerical Aspects

In the following we discuss several problems of Section 4 with respect to their numerical implementation.

Let us consider the elliptic system with distributed control

$$Ru = \sum_{k=1}^{\infty} \mu_k^{-1}(u, e_k)\, e_k, \qquad Y = X = L_2(\Omega).$$

Let the dimension n of the state space be one and $\Omega = (0, 1)$, $y(0) = y(1) = 0$, $z(x) = x$. We calculate upper and lower bounds according to Theorem 5, selecting $\lambda = 1$:

N	d	low. bd.	upp. bd.	diff.
1	0.017677229	0.34883697	0.50237259	0.155
2	0.017954882	0.41406715	0.50147980	0.088
3	0.017980059	0.44032748	0.50140089	0.062
4	0.017984555	0.45446599	0.50138683	0.048
5	0.017985735	0.46329400	0.50138314	0.039
10	0.017986411	0.48176570	0.50138103	0.021
20	0.017986438	0.49142699	0.50138094	0.011
50	0.017986439	0.49736329	0.50138094	0.005

This table shows the monotone behavior of upper and lower bounds and their differences.

In the parabolic initial value control problem with point observation at

$\bar{x} \in \Omega$ the functions $\{\exp(-\mu_k^2 \cdot)\}$ play the role of the system which has to be minimal. As mentioned earlier in the case of dimension $n = 1$, we know [11] that this system is minimal. However, for numerical purposes it is necessary to have the biorthogonal system explicitly, which for an infinite system is a difficult task to solve. However, since the sequence of functions is decreasing rather fast it is advisable to estimate the error by considering only a finite part of the sequence. In this case the computation of a biorthogonal system reduces to an inversion of the Gram's matrix of $\{\exp(-\mu_k^2 \cdot)\}$. For observation of the state at time $T > 0$ with $X = Y = L_2(\Omega)$ we have a similar situation as in the elliptic case discussed before.

Finally we would like to mention some remarks in the case where "nonsmooth" in X and Y are used. In [5] and [9] a problem of parabolic boundary control was considered, where $X = L_\infty(0, T)$ and $Y = C(\Omega)$ with the maximum norm. The numerical results in both papers indicate that it is not advisable to use W as the set of linear combinations of the functions forming the biorthogonal system. On behalf of the special structure of the extreme functional it is more convenient to select some fixed points $w_1, \ldots, w_N \in \Omega$ and to define

$$W = \left\{ l \in C(\Omega)^* : l = \sum_{i=1}^{N} a_i l_i, \text{ where } l_i(f) = f(w_i) \text{ for } f \in C(\Omega) \right\}.$$

On the other hand, if $Y = L_1(\Omega)$, by the extreme functionals we are led to use piecewise continuous functions in W. For instance, if $(0, 1) = \Omega \subseteq \mathbb{R}^1$,

$$W = \left\{ l \in Y^* : l = \sum_{i=1}^{N} a_i \chi_i, \text{ where } \chi_i \text{ is the characteristic function on } [t_{i-1}, t_i) \right\}$$

with a fixed partition $0 = t_0 < t_1 < \cdots < t_N = 1$ of Ω.

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